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A remarkable eight-point planar configuration

Peter C. Fishburn *

AT&T Labs-Research, Information Sciences Research Centre, Room C227, 180 Park Avenue, Florham Park, NJ 07932, USA

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Abstract

There is a unique eight-point planar configuration H_8 in which each point has exactly three distinct distances to the other seven, and it is the only eight-point configuration that minimizes the sum of the points' distance counts. The points in H_8 are the vertices of two same-centered squares at a rotation of 45° with side-lengths ratio $2\cos 15°$. I first heard about H_8 from Paul Erdős, who heard about it from Heiko Harborth. © 2002 Elsevier Science B.V. All rights reserved.

MSC: 52C10; 51K99

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1. Introduction

There is a remarkable eight-point planar configuration H_8 that I first heard about from Paul Erdős, who learned about it from Heiko Harborth. The configuration's points are the vertices of two squares at an angle of 45° to one another that have a common center and side-lengths ratio $2\cos 15° = 1.93$. Each circle centered at a vertex of the larger square whose radius is the side length of that square contains two adjacent vertices of the smaller square, and each inner vertex also has four others equidistant from it. In addition, the configuration includes the vertices of eight equilateral triangles among its three-point subsets, and the perpendicular bisector of the line segment between any two points contains exactly two other points. It is pictured at the top of Fig. 1.

A few definitions will preface a summary of H_8's other special properties within the class of all eight-point planar sets. Let dds mean *different distances*. The *distance count* of a point in a finite planar set is the number of dds from that point to the others. The *distance-count vector* of a planar n-set is the n-tuple $f = f_1 f_2 \ldots f_n$ of its points' distance counts in nondecreasing order, so $1 \leqslant f_1 \leqslant f_2 \leqslant \cdots \leqslant f_n \leqslant n-1$,

* Corresponding author. Tel.: +1-973-360-8413; fax: +1-973-360-8178.
E-mail address: fish@research.att.com (P.C. Fishburn).

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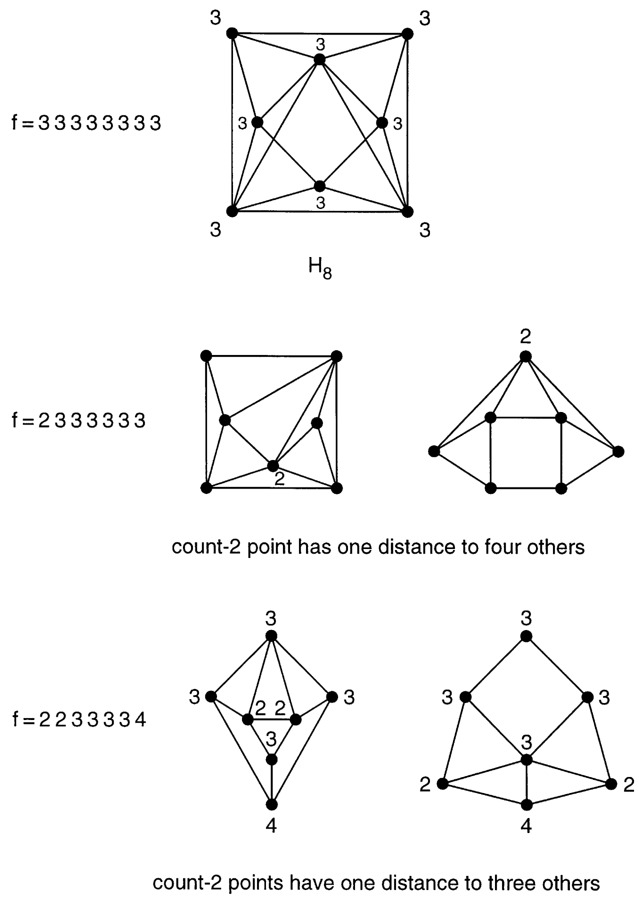

Fig. 1.

and its *distance-count sum* is $f_1+f_2+\cdots+f_n=\sum f_i$. Distance counts are invariant to similarity transformations, and configurations related by similarity transformations will be regarded as identical.

Erdős and Fishburn [2] prove that min $\sum f_i=24$ over all eight-point planar sets and note that this minimum is attained by H_8. They show also for $n=7$ that min $\sum f_i=19$, which is uniquely realized by a regular hexagon's vertices plus its center with $f=1333333$; and that $f=2333333$ is attained only by the two configurations in the middle row of Fig. 1 when the count-2 point has the same distance to exactly four of the other six points. The following theorem strengthens these results with uniqueness assertions for H_8.

Theorem 1. *Each of* P1, P2 *and* P3 *holds for* H_8 *and for no other eight-point planar configuration*:

P1. $f=33333333$;

P2. *The removal of some point* (*in fact, any point*) *leaves a seven-point configuration with* $f=2333333$;

P3. $\sum f_i=24$.

Thus, among all eight-point configurations, H_8 is the unique minimizer of $\sum f_i$ and is the only configuration with $f_i \leqslant 3$ for all i. Moreover, among all seven-point configurations, only the two in the middle row of Fig. 1 have $f=2333333$, with $\sum f_i=20$, and both are subconfigurations of H_8. There are at least three other seven-point configurations with $\sum f_i=20$, but each has $f=2233334$: see Fig. 5 of [2]. Two of these are shown in the bottom row of Fig. 1. They have the property that a count-2 point has the same distance to three other points and another distance to the remaining three. P2 implies that $f=2333333$ can occur only when the count-2 point has the same distance to *four* others.

It is easily seen that $f=2333333$ is impossible if the count-2 point has the same distance to five of the other six points. We complete the proof of P2 in Section 2 by proving

Theorem 2. *If some point in a seven-point configuration has one distance to three other points and another distance to the remaining three, then* $f_i \geqslant 4$ *for some* i.

Section 3 then proves P1.

Theorem 3. H_8 *is the unique eight-point configuration with* $f=33333333$.

These results and results in [2] imply.

Corollary 1. H_8 *is the unique minimizer of* $\sum f_i$ *over all eight-point configurations.*

Proof. Suppose $f_1=1$. Then, by Theorem 2f in [2], $f_2 \geqslant 3$ and $f_3 \geqslant 4$, so $f_1+\cdots+f_8 \geqslant 28$. Suppose $f_1=2$. If $f_2=2$ then, by Theorems 3e and 3h in [2], $\sum f_i \geqslant 26$. If $f_2=3$ when $f_1=2$, $\sum f_i=24$ if and only if $f=23333334$, and this could occur according to Theorems 3d and 3f in [2] only if there is a seven-point subconfiguration with $f=2333333$. P2 of Theorem 1 implies that the subconfiguration must be one of the two in the middle row of Fig. 1. However, when an eighth point is added to either configuration while preserving $f_1=2$, so that the new point lies on one of the two circles centered at the count-2 point that contain the other six, at least two of the others will have distance counts of 4 or more, so in fact $\sum f_i \geqslant 25$. We conclude that $\sum f_i=24$ can occur only if $f_1=3$, and it follows from Theorem 3 that H_8 is the unique minimizer of $\sum f_i$ over all eight-point configurations. □

Section 4 concludes the paper with a brief discussion that highlights a min $\sum f_i$ conjecture for $n=9$.

2. Proof of Theorem 2

To prove Theorem 2, let $\{x, 1, 2, 3, 4, 5, 6\}$ be a seven-point planar configuration such that x has distance count 2, points 1, 2 and 3 lie on a circle centered at x, and $4, 5$ and 6 lie on a larger circle centered at x. Also let g_i be the distance count of point $i \geqslant 1$ in $\{x, 1, 2, \ldots, 6\}$. We try to position points 1–6 so that each has $g_i \leqslant 3$, and will find that this is impossible. Our working hypothesis, which will ultimately be contradicted, is $g_i \leqslant 3$ for $i = 1, \ldots, 6$.

The focus will be on configurations of $\{x, 4, 5, 6\}$ and how $1, 2$ and 3 might be positioned on a smaller circle centered at x. For each $i \in \{4, 5, 6\}$, let h_i be the distance count of i within $\{x, 4, 5, 6\}$, so $h_i \in \{1, 2, 3\}$. Also let B^* denote the set of perpendicular bisectors of the six line segments between distinct points in $\{x, 4, 5, 6\}$.

Lemma 1. *Suppose $g_i \leqslant 3$ for $i = 1, \ldots, 6$. Then every point in $\{1, 2, 3\}$ lies on a line in B^*. If $h_i = 2$ for an $i \in \{4, 5, 6\}$, then at least one point in $\{1, 2, 3\}$ lies on a circle centered at i whose radius is one of the two dds from i to the others in $\{x, 4, 5, 6\}$. If $h_i = 3$ for an $i \in \{4, 5, 6\}$, then all points in $\{1, 2, 3\}$ lie on the circles centered at i whose radii are the dds from i to the others in $\{x, 4, 5, 6\}$. (All points in $\{1, 2, 3\}$ also lie on one circle centered at x that is inside the circle of* 4, 5 *and* 6.)

Proof. If point 1 is not on a B^* line, it has dds to all points in $\{x, 4, 5, 6\}$ and $g_1 \geqslant 4$. Since point 4 cannot have the same distance to all points in $\{1, 2, 3\}$ (nonidentical circles intersect in at most two points), $h_4 = 2$ and $g_4 \leqslant 3$ imply that the distance between 4 and some point in $\{1, 2, 3\}$ duplicates a distance between 4 and some point in $\{x, 5, 6\}$. If $h_4 = 3 = g_4$, the dds between 4 and the points in $\{1, 2, 3\}$ must be a subset of the dds between 4 and the points in $\{x, 5, 6\}$. □

The proof of Theorem 2 now follows the dictates of Lemma 1 and divides into cases. Fig. 2 pictures the four possibilities for $\max\{h_i \colon i \in \{4, 5, 6\}\} = 2$ (see, e.g., Theorems 2b and 2c in [2]), where the solid lines above the bottom blow-up are the perpendicular bisectors in B^*. The vertices of equilateral triangles in the figure are $45x$ and $56x$ for I, 456 for II, and $46x$ for III and IV. We comment on I–IV and then consider other cases for $\{x, 4, 5, 6\}$.

I. $h_4 = h_6 = 2$. By Lemma 1, we assume without loss of generality that point 1 is one of the three $\circ$ points and 2 is one of the three $*$ points. Since 1 and 2 are on the same circle centered at x, they lie on the same horizontal line, so there are three choices for 1 and 2: top, middle, or bottom $\circ$ and $*$. In the bottom choice, 1 has increasing dds to 2, 5, 6 and 4, so $g_1 \geqslant 4$. For the top and middle choices, $g_1 \leqslant 3$ and $g_2 \leqslant 3$ each allows three places for point 3 on the circle centered at x that contains 1 and 2 (to duplicate one of the shorter distances from 1 or 2 to the others), but the three-place sets are disjoint and force $g_1 = 4$ or $g_2 = 4$.

II. $h_4 = h_5 = h_6 = 2$. Here, and later, *let rs denote the distance between points r and s*. The circle centered at 4 with radius $4x$ intersects no member of B^* inside the

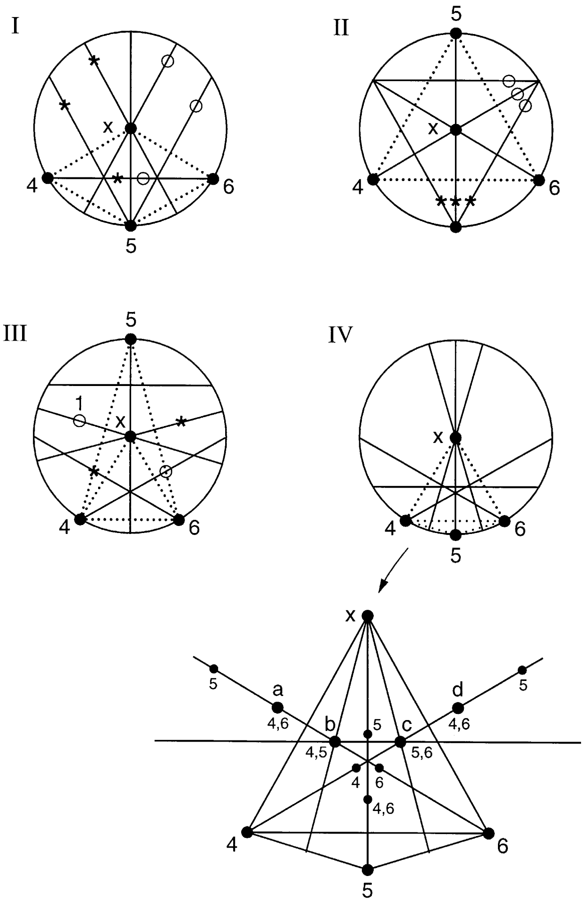

Fig. 2.

displayed circle except at x, so we may assume that point 1 is one of the three $\circ$ points with $41=45$. Similarly, point 2 is one of the three $*$ points, and point 3 has three possibilities to the northwest of x. If 1 is the top $\circ$, it has dds to 5, 6, 2 and 4; if 1 is the middle $\circ$, it has dds to x, 6, 2 and 4. It follows from symmetry that $g_i \geqslant 4$ for $i=1,2,3$.

III. $h_4=h_5=h_6=2$. The circle centered at 4 with radius 45 does not intersect a member of B^* inside the displayed circle, so we assume that point 1 is one of the two $\circ$ points with $41=4x$. Similarly, point 2 with $62=6x$ is one of the two $*$ points. The $\circ$ and $*$ points lie on the same circle C centered at x, so 3 is also on this circle at one of its B^* intersections. Since $h_5=2$, we assume without loss of generality that 1 is the upper left $\circ$ point. If 2 is the upper right $*$ then 3 must be a C where it intersects the vertical through 5 above the solid horizontal line since this is the only place for $g_4 \leqslant 3$ and $g_6 \leqslant 3$; but then $g_3=4$. If the upper right $*$ is not used, then the lower $\circ$ and $*$ points must be used to satisfy $g_5 \leqslant 3$; but then $g_6=4$.

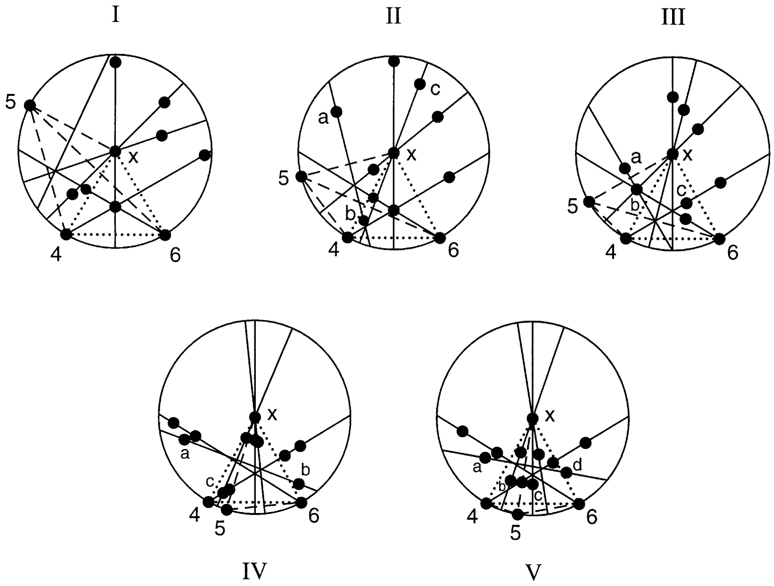

Fig. 3.

IV. $h_4 = h_5 = h_6 = 2$. The 10 newly labeled points in the bottom blow-up for IV are the B^* points that might be used for 1, 2 and 3 according to the $h_i = 2$ part of Lemma 1. An i label indicates a duplicated distance from point i to another in $\{x, 4, 5, 6\}$. The circle centered at x that contains 1,2 and 3 must contain newly labeled points whose labels cover $\{4, 5, 6\}$. The only circle that does this is the one through a, b, c and d, and, for coverage of $\{4, 5, 6\}$, we assume that one of $\{a, b\}$, $\{a, c\}$ and $\{b, c\}$ is a subset of $\{1, 2, 3\}$. If $\{a, b\}$ is used, $g_a \geqslant 4$. (Symmetrically, $\{c, a\}$ forces $g_d \geqslant 4$.) If $\{a, c\}$ is used, $g_5 = 3$ also forces the use of b or d, so some $g_i \geqslant 4$. Finally, if $\{b, c\}$ is used, $g_4 = 3$ forces the use of a or d, so again some $g_i \geqslant 4$.

This completes our analysis for Fig. 2, so we assume henceforth that $h_i = 3$ for some $i \in \{4, 5, 6\}$.

We suppose next that $4, x$ and 6 are the vertices of an equilateral triangle and $h_5 = 3$. Fig. 3 shows positions for 5, with two special cases: in III, the angle between $[4, x]$ and $[5, x]$ is 30°; in V, the same angle is 20°. The B^* lines are solid, and the solid interior points are the intersections of those lines with the circles centered at 5 whose radii are 54, $5x$ and 56.

Let C denote a circle centered at x through three or more solid interior points. Fig. 3I has no C, and II, III and IV each has one C which contains points a and b on the perpendicular bisector of $[5, x]$, and point c on another line in B^*. In each case, $\{a, b\} \subset \{1, 2, 3\}$, but a has 4 dds to $\{4, 5, 6, b\}$. Fig. 3V has one C; it contains a and d on the perpendicular bisector of $[5, x]$, and b and c on the perpendicular bisectors of $[4, 5]$ and $[4, 6]$. (This is the only case of more than three points in C, and no case

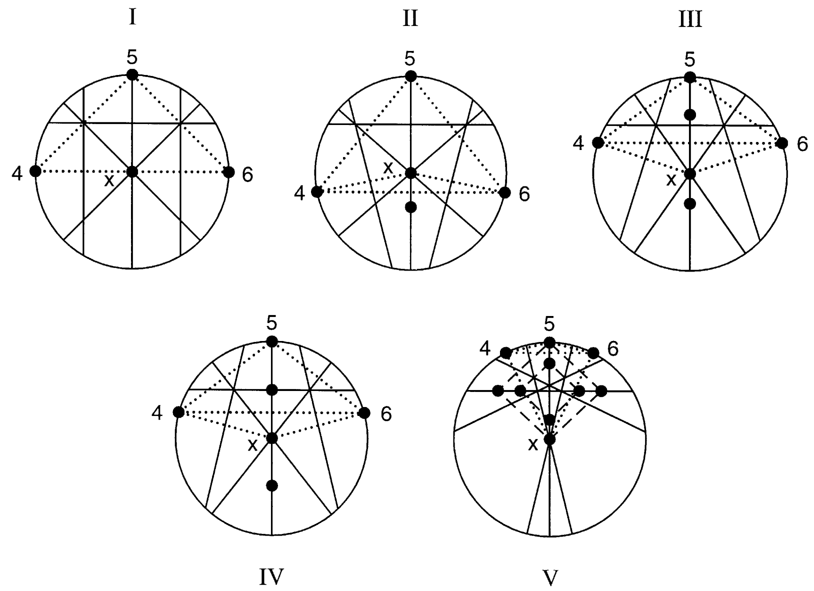

Fig. 4.

has more than one C.) We require $\{1,2,3\} \subset \{a,b,c,d\}$, but a has four dds to 4, 5, 6 and any one of b,c, and d, while $g_d \geqslant 4$ in $\{x,4,5,6,b,c,d\}$.

We assume henceforth that $\{x,4,5,6\}$ has no equilateral triangle. The only remaining cases that have $h_i < 3$ for some i are pictured in Fig. 4, where $h_5 = 2$ and $h_4 = h_6 = 3$. The solid interior points are those on lines in B^* which duplicate a distance in $\{4x,45,46\}(=\{6x,65,64\})$ from 4 and from 6. The top three cases have 0, 1 and 2 such points besides x. Most other specific placements of 4 and 6 are similar to II and III, with no circle centered at x containing more than one such point. Exceptions are seen in IV, where $[4,6]$ is one-fourth of the way from x to 5, and in V, which has four solid interior points on the perpendicular bisector of $[5,x]$ and features two squares centered at the midpoint of $[5,x]$. In V, the lines from that midpoint to 4 and 6 make angles of 45° with the line from the midpoint up to 5. Cases IV and V have circles centered at x that contain two solid interior points, but no case has more than two.

We assume henceforth that $h_i = 3$ for $i = 4,5,6$. Let $\alpha = 4x = 5x = 6x,\ \beta = 45,\ \gamma = 46,\ \delta = 56$ with $|\{\alpha,\beta,\gamma,\delta\}| = 4$. For visual convenience, we continue to place 4 and 6 on a horizontal line with 4 left of 6. The supposition that $g_i \leqslant 3$ for all i requires

$$\{41,42,43\} \subseteq \{\alpha,\beta,\gamma\},$$

$$\{51,52,53\} \subseteq \{\alpha,\beta,\delta\},$$

$$\{61,62,63\} \subseteq \{\alpha,\gamma,\delta\}.$$

The following lemma leads to an impossibility for this final case.

Lemma 2. *Suppose* $h_i = 3$ *for* $i = 4, 5, 6$, *and* $g_i \leqslant 3$ *for* $i = 1, \ldots, 6$. *Then* $\{41, 61\}$ *is not a subset of* $\{\alpha, \gamma\}$.

We defer the proof until after we show how the lemma produces a contradiction. Given Lemma 2, it follows from notational permutations that, for every $i \in \{1, 2, 3\}$,

(i) $\{4i, 6i\} \nsubseteq \{\alpha, \gamma\}$,
(ii) $\{4i, 5i\} \nsubseteq \{\alpha, \beta\}$,
(iii) $\{5i, 6i\} \nsubseteq \{\alpha, \delta\}$.

Suppose $41 = \alpha$. Then (i) $\Rightarrow 61 = \delta$ and (ii) $\Rightarrow 51 = \delta$, which contradict (iii). Hence, $41 \neq \alpha$. Similarly, no ki for $k \geqslant 4$ and $i \leqslant 3$ equals α. Therefore,

$$\{41, 42, 43\} \subseteq \{\beta, \gamma\},$$

$$\{51, 52, 53\} \subseteq \{\beta, \delta\},$$

$$\{61, 62, 63\} \subseteq \{\gamma, \delta\}.$$

Since $ji \neq ki$ for distinct j and $k \geqslant 4$ and $i \leqslant 3$, by (i)–(iii), each $(4i, 5i, 6i)$ for $i = 1, 2, 3$ must be (β, δ, γ) or (γ, β, δ), and therefore at least two of the three $(4i, 5i, 6i)$ are identical. Suppose with no loss of generality that $(41, 51, 61) = (42, 52, 62)$. Then 4, 5 and 6 all lie on the perpendicular bisector of [1,2], which is impossible because 4, 5 and 6 are assumed to lie on a circle.

The proof of Lemma 2 has three parts, for $41 = 61 = \alpha$, $41 = 61 = \gamma$ and $\{41, 61\} = \{\alpha, \gamma\}$. We show how each of these yields a contradiction to $g_i \leqslant 3$ for all i.

Part 1: Suppose $41 = 61 = \alpha$. Point 5 must be on a perpendicular bisector for a pair in $\{x, 1, 4, 6\}$, and the only feasible pairs at this point are $\{1, 4\}$ and $\{1, 6\}$. Assume for definiteness that 5 is on the perpendicular bisector of $[1, 6]$. The top row of Fig. 5 illustrates the two basic possibilities. In both cases, 1 has dds to $x, 4$ and 5 (except for a special case noted in the final paragraph of this section), and 2 and 3 must be on the circle through 1 centered at x. In the upper-left case, 2 and 3 must be on or below the dashed circular arc at distance α from 1, and when we consider distances to 2 and 3 from the others, e.g. 4 and 6, the hypothesis that $g_i \leqslant 3$ for all i shows that there are no acceptable positions for 2 and 3. In the upper-right case, 2 and 3 must be on or below the dashed circular arc there, and consideration of distances from 1 and 4 shows that $g_i \leqslant 3$ for all i is impossible.

Part 2: Suppose $41 = 61 = \gamma = 46$. Here 5 must be on the perpendicular bisector of $[1, x]$, $[1, 6]$ or $[1, 4]$. The two prototypes of $[1, x]$ are shown in the middle row of Fig. 5. We have $\mu + \tau + \theta = 180^\circ$ for the left case, and $\mu + \tau + \theta = 90^\circ$ for the right case. In both cases, $\theta = 60^\circ$, and this implies that $56 = \alpha$ so that $x, 5$ and 6 are the vertices of an equilateral triangle, contrary to $h_i = 3$ for $i = 5, 6$. The same contradiction obtains if 5 is on the perpendicular bisector of $[1, 6]$, for then 5 is equidistant from 1 and 6, 4 is equidistant from 1 and 6, and the result for $[1, x]$ implies that 5 is the same place as before.

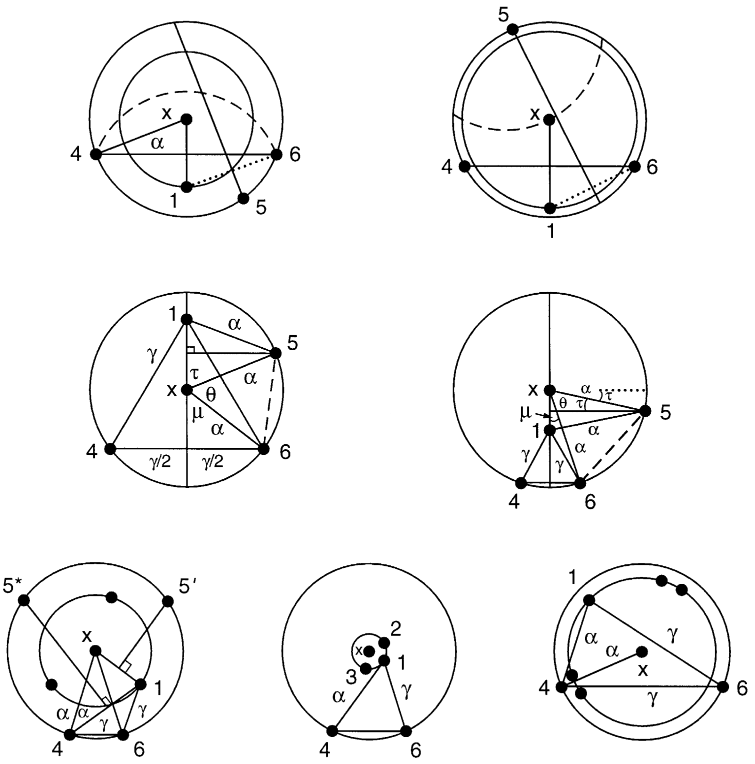

Fig. 5.

It remains to verify that $\theta = 60^\circ$. Let $\rho = (\gamma/2)/\alpha$. For the left diagram, pythagorean calculations show that $\sin\mu = \rho$ and $\cos\tau = (\sqrt{3}/2)\rho - \sqrt{1-\rho^2}/2$. Therefore,

$$\cos\tau = \sin 120_\circ \sin\mu + \cos 120_\circ \cos\mu$$

$$= \cos(120_\circ - \mu),$$

so that $\tau = 120^\circ - \mu$, or $\theta = 180^\circ - (\tau + \mu) = 60^\circ$. For the right diagram, we have $\sin\mu = \rho$ and

$$\sin\tau = \frac{1}{2}\sqrt{1-\rho^2} - \frac{\sqrt{3}}{2}\rho$$

$$= \sin 30_\circ \cos\mu - \cos 30_\circ \sin\mu$$

$$= \sin(30_\circ - \mu),$$

so $\tau = 30^\circ - \mu$ and $\theta = 90^\circ - (\tau + \mu) = 60^\circ$.

Part 3: Suppose $41 = \alpha$ and $61 = \gamma$, as illustrated in the bottom row of Fig. 5. Except for the unique case shown at the left, point 1 has dds to $x, 4$ and 6. The right two diagrams show possibilities for 2 and 3 on the circle through 1 centered at x, according to distances allowed from point 1. In the central diagram, 2 has four dds to the others.

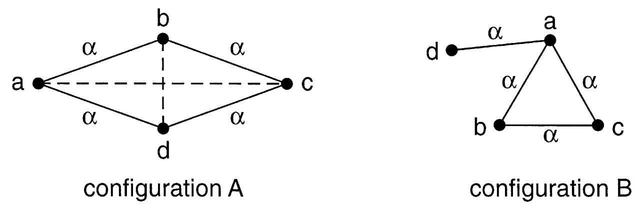

Fig. 6.

On the right diagram, every choice of two of the four unlabeled points for 2 and 3 gives a point with four or five dds to the others.

Points x, 1, 6 and 4 on the lower left diagram of Fig. 5 are four of the five vertices of a regular pentagon. Point 5 must be on a perpendicular bisector of $[1, y]$ for some $y \in \{x, 4, 6\}$, and because the perpendicular bisectors of $[1, 6]$ and $[x, 4]$ are identical, only the points marked 5^* and $5'$ are suitable for 5. If 5^* obtains, the only plausible positions for 2 and 3 are the two unlabeled points at distance $x5^* = \alpha$ from 5^*. But the upper unlabeled point has at least four dds to the others. If $5'$ obtains, we have a situation similar to the upper left diagram of Fig. 5. In this case, consideration of allowed distances from points 4 and $5'$ shows that the only feasible point for 2 and 3 is the lower unlabeled point, so 2 and 3 cannot be specified to give $g_i \leqslant 3$ for all i.

3. Proof of Theorem 3

Assume that $f = 33333333$ for $X = \{1, 2, \ldots, 8\}$. Suppose X contains distinct i and j such that $ij \neq ik$ for all $k \in X \setminus \{i, j\}$. Then, when j is removed, we get a seven-point configuration with at least one distance count 2. By previous results, the distance-count vector for $X \setminus \{j\}$ must be 2333333. It then follows from P2 of Theorem 1 that $X = H_8$.

Suppose $X \neq H_8$. Then every $i \in X$ has *multiplicity vector* $(2, 2, 3)$: i has one distance to two other points, a second distance to two more, and a third distance to the remaining three. We will see that this is impossible, so H_8 is the only eight-point configuration with $f = 33333333$.

We suppose henceforth that every point in X has multiplicity vector (2,2,3) and refer to the distance which occurs 3 times from i to the other points as i's *plurality distance*. To show impossibility, we begin with a combinatorial lemma based on two four-point configurations. Four points a, b, c and d have *configuration A* if (for some labeling) $ab = bc = cd = da$, and have *configuration B* if $ab = bc = ca = ad$: see Fig. 6.

Lemma 3. *If $X \neq H_8$ and X has $f = 33333333$, then X contains four points that have configuration A or B.*

Proof. Given the hypotheses, every point has multiplicity vector (2,2,3). We suppose that X has no A or B configuration and obtain a contradiction.

Assume without loss of generality that 1's plurality distance is α with $12=13=14=\alpha$. To avoid configuration B, none of 23, 34 and 24 is α. Since each of 2, 3 and 4 requires distance α to some point other than 1, assume that $2j_2=3j_3=4j_4$ for some $j_2,j_3,j_4\in\{5,6,7,8\}$. To avoid configuration A (with 1, two from $\{2,3,4\}$, one from $\{5,6,7,8\}$), we require $|\{j_2,j_3,j_4\}|=3$ and assume without loss of generality that $(j_2,j_3,j_4)=(5,6,7)$, with none of 26, 27, 35, 37, 45 and 46 equal to α. We have

$$12=13=14=25=36=47=\alpha,$$

with no other α's within $\{1,\dots,7\}$ except perhaps for 56, 67 and 57. At this point, another α is needed for each of 5, 6 and 7. If point 8 has no α distance to another point, we use α for any two of 56, 67 and 57, but not all three because of configuration B. This gives exactly eight instances of α, which is the plurality distance of exactly two points, 1 and one of 5, 6 and 7.

Two main cases arise if $\alpha=i8$ for two or three $i\geqslant 2$. The first has $\alpha\in\{28,38,48\}$, say $\alpha=28$ without loss of generality, which requires $\alpha\notin\{38,48\}$ to avoid configuration A. We then need another α for each of 5, 6, 7 and 8. This can be done with two more α's, say $\alpha=58=67$, which gives nine instances of α and two plurality distances, for 1 and 2. It can also be done with three more α's, say $\alpha=56=67=78$ without loss of generality. This gives 10 occurrences of α, which is then the plurality distance of four points, namely 1, 2, 6 and 7.

The second main case for 8 has $\alpha\notin\{28,38,48\}$. Here we take $\alpha=58=68$ without loss of generality to provide 8 with two instances of α. We then take $\alpha=67$ or $\alpha=78$, but not both because of configuration B, to obtain a second α for 7. This gives nine occurrences of α and two plurality distances, for 1 and 6 or 1 and 8.

The following list summarizes the possibilities when α is the plurality distance of point 1

(i) no α for 8: 8 α's, two plurality distances,
(ii) $\alpha=28$: 9 α's, two plurality distances, or
(iii) $\alpha=28$: 10 α's, four plurality distances,
(iv) $\alpha\notin\{28,38,48\}$: 9 α's, two plurality distances.

In addition, we note that there are $\binom{8}{2}=28$ pairs in $\{\{i,j\}: i,j\in X, i\neq j\}$, every eight-point planar configuration determines at least four dds (see [1] or [3]), and each of the eight points in X has one and only one plurality distance for the 3 in its multiplicity vector (2,2,3).

It follows that exactly two or three dds are needed to produce the plurality distances for the eight points in X. If three are used, say α, β and γ, we require one to be similar to (iii) with plurality coverage for four points, and the others to be similar to (i), (ii) or (iv) with plurality coverage for two points each. However, this requires at least $10+8+8=26$ occurrences of α, β and γ, leaving at most two $\{i,j\}$ for the necessary fourth distance and giving a contradiction to $(2,2,3)$ as the multiplicity vector of every point.

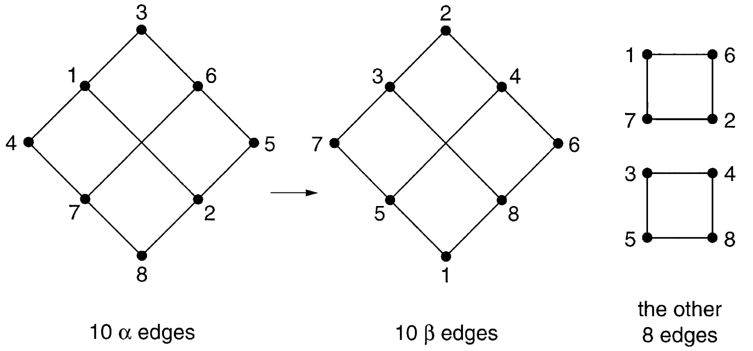

Fig. 7.

We conclude that exactly two distances, say α and β, give plurality distances for all eight points. Each must be similar to (iii) with 10 instances and plurality coverage for four of the eight points. They leave eight $\{i,j\}$ for other distances, each of which must have zero or two instances from any point to the others.

Since our analysis for α was unique up to permutations on the points, we assume without loss of generality that

$$\alpha = 12 = 13 = 14 = 25 = 36 = 47 = 28 = 56 = 67 = 78$$

The α edges within the complete graph on X are shown at the left of Fig. 7. The plurality distances α points are 1, 2, 6 and 7, so those for β must be 3, 4, 5 and 8. All β edges must differ from the α edges. A feasible realization of β appears in the middle of the figure: the eight remaining edges are on the right. The only way to assign other distances to those eight that preserve the multiplicity vectors $(2,2,3)$ is $\gamma = 16 = 26 = 27 = 17$ and $\delta = 34 = 48 = 58 = 35$, and these force configuration A.

The β of Fig. 7 gives the final contradiction to our supposition that X has no A or B configuration. Exactly the same contradiction occurs for every feasible β. In going from α to β, $\{1,2,6,7\}$ must map onto $\{3,4,5,8\}$, and edge $\{1,2\}$ and $\{6,7\}$ for α must map onto edges $\{3,8\}$ and $\{4,5\}$ for β. If, for example, 3 and 8 are not adjacent for β, they will have one intermediate vertex from $\{1,2,6,7\}$. However, $\{1,3\}$, $\{3,6\}$, $\{2,8\}$ and $\{7,8\}$ are α edges, so there is no way to label the β vertex between 3 and 8 to avoid duplication of an α edge. Given any feasible placement of $\{3,4,5,8\}$ for β, there is exactly one way to label the other four to avoid an α duplication, and every β thus constructed leaves the eight edges shown at the right of Fig. 7. $\square$

The proof of Theorem 3 will be completed by proving the antithesis of Lemma 3.

Lemma 4. *If* $X \neq H_8$ *and* X *has* $f = 33333333$, *then* X *cannot contain four points that have configuration* A *or* B.

Proof. Assume the hypotheses, so every point in X has multiplicity vector $(2,2,3)$. Let $Y = \{1,2,3,4\}$ be the point set for a supposed A or B configuration, and let B^* be

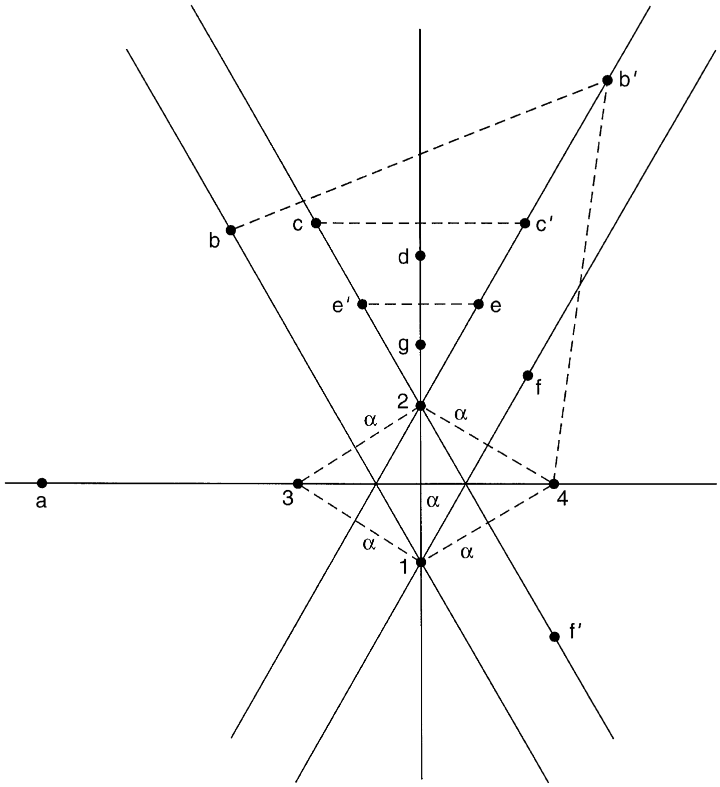

Fig. 8.

the set of perpendicular bisectors of the line segments joining points in Y. As usual, we presume that 5–8 lie on lines in B^*. We begin by showing impossibility for the four A and B configurations that have only two dds.

Case 1: See Fig. 8. Let $\beta = 34$. Assume $\beta = 35$ for a second instance of β from 3. The positions of 5 on the B^* lines at distance β from 3 are a through f (up to top-bottom symmetry). Each has three dds to Y. There is no point besides 3 and a point to the left of a on the B^* lines at distance $a3$ from a, and the point left of a has four dds to $\{a, 3, 2, 4\}$, so a cannot be used for 5. For b, we duplicate $b4$ from b and find that b' is the only B^* point with $bb' = b4$ for which no point in $Y \cup \{b, b'\}$ has more than three dds to the others. We return to this subcase in the next paragraph. For c, we duplicate $c2$ and obtain c' as the only point with $cc' = c2$ for which no point in $Y \cup \{c, c'\}$ has more than three dds to the others. Point $d \neq 5$ because it gives a fourth instance of α from 2. For e, we duplicate $e2$ and obtain e' as the only point with $ee' = e2$ for which no point in $Y \cup \{e, e'\}$ has more than three dds to the others. And for f, we duplicate $f3$ and obtain c and f' as the only B^* points with fewer than four dds to the others. However, 4 has four dds to $\{1, 2, 3, f, c\}$, and $1f' = \alpha$ for four instances of α from 1, so $f \neq 5$.

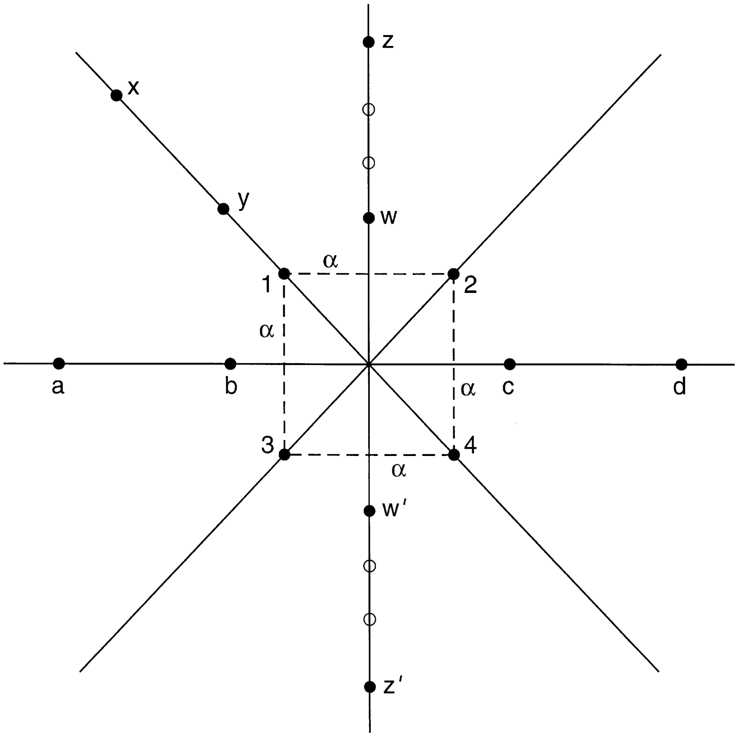

Fig. 9.

The subcases for Case 1 that require further scrutiny are $(5,6)=(b,b')$, $(5,6)=(c,c')$ and $(5,6)=(e,e')$. For (b,b'), we duplicate $b'2$ from b' and find that every such point on B^* has four dds from it to the others or from 3 to the others, so $(5,6)\neq(b,b')$. For (c,c'), duplication of $c'4$ from c' gives no point on B^* lines with fewer than four dds to $Y\cup\{c,c'\}$, so $(5,6)\neq(c,c')$. For (e,e'), the only point on B^* with distance $4e$ from 4 that has only three dds to the others is g, but e has dds to g, 2, 4 and 1, so $(5,6)\neq(e,e)$. This completes our analysis of Case 1.

Case 2: See Fig. 9. Let $\beta=14=23$. Points x and y on the perpendicular bisector of [2,3] have $x1=\beta$ and $y2=\beta$ with three dds to Y, but neither can be used as a point in $\{5,6,7,8\}$ because their only other point on B^* with the same distance to 1 lies northwest of x or y and has four dds to $Y\cup\{x \text{ or } y\}$. Hence, points for second β distances from 1, 2, 3 and 4 must be on the horizontal or vertical members of B^*, like a, b, w and z. Suppose point 5 is a with $a1=a3=\beta$. Then, to obtain second β distances for 2 and 4, we must use b, or d, or, if neither b or d, one of the following pairs: $\{z,w\}$, $\{z',w'\}$, $\{z,z'\}$, $\{w,w'\}$. However, if such a pair is used, one of its members has four dds to the others. A similar conclusion holds if we suppose that $5=b$ instead of $5=a$. It follows that we can assume without loss of generality that $(5,6)$ is (a,b) or (a,d) or (b,c).

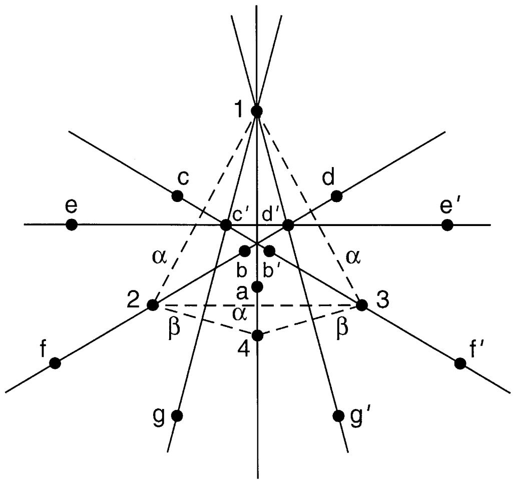

Fig. 10.

We have $(5,6) \neq (a,b)$ because the only point on B^* at distance ab from a is to the left of a and has four dds to $\{a,b,1,2\}$. If $(5,6)=(b,c)$, the only points on B^* that duplicate one of the three established dds for b *and* one of the three for c are the four marked as $\circ$ on the vertical line. Any two of those four in union with $Y \cup \{b,c\}$ give a point with four dds to the others, so $(5,6) \neq (b,c)$. A similar result obtains if $(5,6)=(a,d)$ is presumed. We conclude that if an eight-point configuration has multiplicity vectors $(2,2,3)$ and contains a square, then $f_i \geqslant 4$ for some i.

Case 3: See Fig. 10. Let $\beta = 24 = 34$, with α for the other pairs in Y. All potential placements on B^* for points that duplicate the β distance from 2 and from 3 are shown as $a,b,b',\ldots,g,g'$. Point a is infeasible because, in trying to get a second $1a$ distance from 1, we find that no intersection of the circle centered at 1 with radius $1a$ and the three circles centered at 4 with radii $4a$, 42 and 41, lies on B^*, except of course a. If b is used for $2b=\beta$, the only second point with distance $1b$ from 1 that lies on B^* and one of the three circles centered at 4 with radii $4b$, 42 and 41 is b', which satisfies $3b'=\beta$. However, b has four dds to $Y \cup \{b'\}$, so b and b' are infeasible. Points e and e' are infeasible because either gives a fourth α from 1; f and f' cannot be used because f has four dds to 2, 4, 3 and f'; g and g' are infeasible because they give a total of four β instances from 4.

This leaves c, c', d and d' for further consideration. All are equidistant from 1. We consider $5 \in \{c,c'\}$ and $6 \in \{d,d'\}$. Since $2,c',d',3$ are equidistant from 4, $(5,6) \neq (c',d')$, and $(5,6) \neq (c,d')$ because $1,c,4,3$ are equidistant from d'. Moreover, $(5,6) \neq (c,d)$ because the three-distance circles centered at 4, c and d jointly intersect only at 1, 2 and 3: any seventh point forces 4, c or d to have four dds to the others. It follows that every eight-point configuration with multiplicities $(2,2,3)$ that has the Y subconfiguration of Fig. 10 also has $f_i \geqslant 4$ for some i.

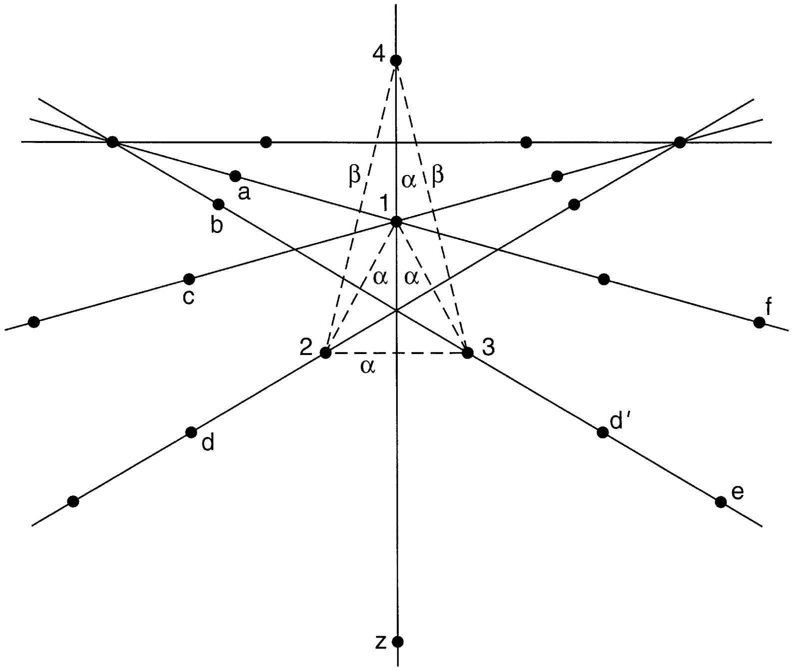

Fig. 11.

Case 4: See Fig. 11. Let $\beta = 24 = 34$, with α otherwise in Y. The solid non-Y dots on the figure are the points on B^* at distance β from 2 or 3. Apart from z at the bottom, where $\beta = 2z = 3z$, we need at least one of the points at distance β from 2, and one at distance β from 3. We first discard the four points on the perpendicular bisector of $[1,4]$ because each inner point creates a fourth instance of α from 1, and each outer point gives a Case 3 configuration with 1, 4, and 2 or 3. Suppose $z \in \{5,6,7,8\}$. When we consider the intersection points of the three circles centered at z with radii $z2$, $z1$ and $z4$, and the three centered at 4 with radii 41, 42 and $4z$, the only intersections that lie on B^* are two on the lower part of the bisector of $[2,4]$ and the symmetric two on the lower part of the bisector of $[3,4]$. The choice of any three of these four to complex X forces $f_i \geqslant 4$ for some i. So we discard z.

The points that remain with distance β from 3 are $a, b, \ldots, f$. Each of these, except d, when paired with any point remaining at distance β from 2, gives either 4 dds from the labeled point to the others or from the latter (distance β from 2) point to the others. The exception is $\{d, d'\}$, where no point $Y \cup \{d, d'\}$ has more than three dds to the others. We then require a seventh point at distance $d2 = \alpha$ from d, an eight point at distance α from d', and note that there is only point 1 at distance α from 4, contrary to multiplicity vector $(2,2,3)$ for 4.

This completes our analysis of Cases 1–4. The hypotheses of Lemma 4 imply that no four points of X have the specializations of configurations A and B shown in Figs. 8–11.

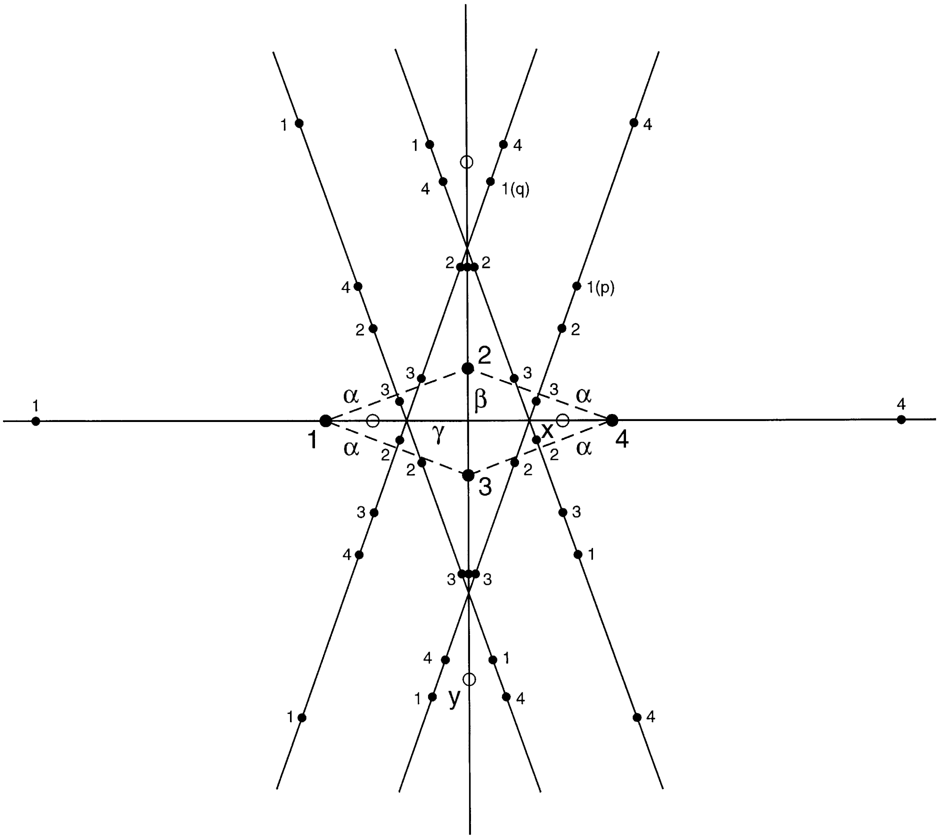

Fig. 12.

We now consider other situations for the two configurations, beginning with A. Fig. 12 pictures A with $\beta = 23$, $\gamma = 14$ and $\beta < \alpha < \gamma$. The other main version of configuration A with three dds has $\alpha < \beta < \gamma$, but its analysis is similar to the ensuring analysis for Fig. 12 and will be omitted. The figure shows all points on B^* that are distance β from 2 or 3, and are distance γ from 1 or 4, labeled by the vertex (large numeral) that they are the indicated distance from. Each $\circ$ on the horizontal is distance β from both 2 and 3, and each $\circ$ on the vertical is distance γ from 1 and 4. We cannot use both horizontal $\circ$ or both vertical $\circ$ because of the four dds proscription, or because they give instances of Case 1. If no $\circ$ is used for a point in $\{5,6,7,8\}$ then the $(2,2,3)$ multiplicities require the use of one small-k point for each $k \in Y$.

Consider $x = \circ$ near 4 for a point in $\{5,6,7,8\}$, with $\beta = x2 = x3$. Point x has dds to 4, 2 and 1, so the others in $\{5,6,7,8\}$ must be on circles centered at x with radii $x4$, $x2$ and $x1$. However, none of those circles pass through a small-4 point, so x is infeasible for the $(2,2,3)$ multiplicities with $f_i \leqslant 3$ for all i.

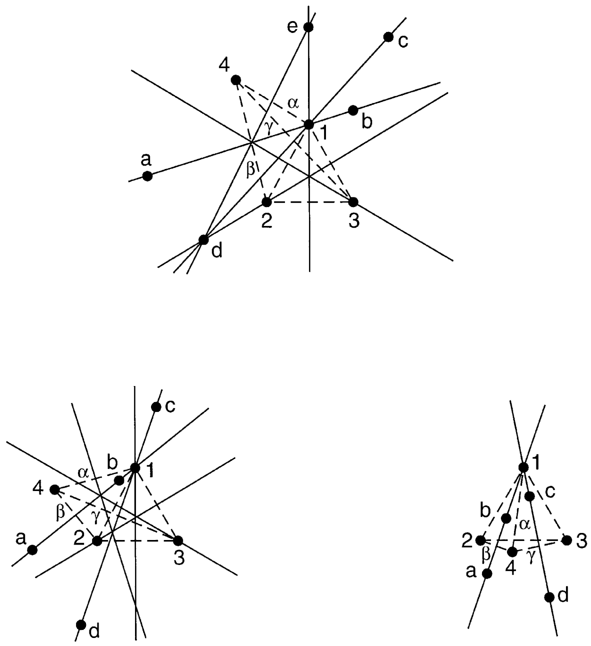

Fig. 13.

Consider $y=\circ$ on the lower vertical for a point in $\{5,6,7,8\}$ with $\gamma=y1=y4$. It has dds to 3, 1 and 2 (if $y1=y2$, see Case 3) so the others in $\{5,6,7,8\}$ must be on circles centered at y with radii $y3$, $y1$ and $y2$. However, none of these go through a small-2 point, so y is infeasible.

With no $\circ$ point in $\{5,6,7,8\}$, consider the circles centered at vertex 4 with radii $\gamma=41$, $\alpha=42$, and δ for the third distance from 4 to other points in X. The circles with radii γ and α miss all small-k points for $k=1,2,3$. (If $4p=\alpha$, see Case 4.) The radius-δ circle must therefore pass through at least one small-k point for every $k\in\{1,2,3\}$. The only conceivable small-1 points for this are those labeled p and q, or their reflections around the horizontal through 1 and 4. Each of p and q already has three dds to Y, and if it were possible to have a radius $4p$ or $4q$ circle centered at 4 which passes through small-k points for $k\in\{2,3\}$, at least one new distance from p or q would occur for those points.

This completes the proof of Lemma 4 for configuration A. We suppose henceforth that configuration B occurs with $\beta=24$, $\gamma=34$, $\beta<\gamma$, and α otherwise for pairs from Y. Three versions are shown in Fig. 13 for different placements of 4. In all cases we look for the places on B^* intersected by the circles centered at 4 of radii α, β and γ which have distance β to 2 for a second instance of β from 2, or which have distance γ to 3 for a second instance of γ from 3. Most cases have only two such points for β, denoted by a and b, and two such instances for γ, denoted by c and d. In the top drawing, where angle 412 is 90°, e is at distance α from 4 and γ from 3, but

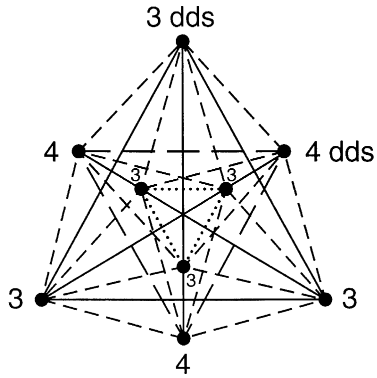

Fig. 14.

$\{1,2,3,e\}$ realizes Case 4. In the lower right drawing, where 412 is approximately 22°, $2b=bc=c1$.

We assume without loss of generality in all cases that $5\in\{a,b\}$ and $6\in\{c,d\}$. In the top special case, $\{d,4,1,3\}$ realizes Case 4, but even so a has four dds to $Y\cup\{a,c\}$ and b has four dds to $Y\cup\{b,d\}$. For the other two $\{5,6\}$ possibilities, a has four dds to $Y\cup\{a,c\}$, and c has four dds to $Y\cup\{b,c\}$. The lower-left case is similar: a has four dds to $Y\cup\{a,c\}$ and to $Y\cup\{a,d\}$, c has four dds to $Y\cup\{b,c\}$, and d has four dds to $Y\cup\{b,d\}$. In the special lower-right case, c has four dds to $Y\cup\{a,c\}$, a has four dds to $Y\cup\{a,d\}$, b has four dds to $Y\cup\{b,d\}$, but no point in $Y\cup\{b,c\}$ has more than three dds to the others. The impossibility of a satisfactory eight-point extension of $Y\cup\{b,c\}$ for the last case follows easily from the further requirements for $\{7,8\}$: one of these two must be distance $1c$ from 1, and the other distance $1b$ from 1; one of 7 and 8 must be distance 41 from 4; one of 7 and 8 must be distance $2c$ from 2; and so forth. We conclude that no four points of an X that satisfies the hypotheses of Lemma 4 can have configuration B.

4. Conjectures

Let $L_\Delta=\{a(1,0)+b(1/2,\sqrt{3}/2)\colon a,b\in\mathbb{Z}\}$, the usual hexagonal lattice. It was conjectured in [2] that all minimizers of $\sum f_i$ for large n are subsets of L_Δ.

We proved in [2] that a subset of L_Δ uniquely minimizes $\sum f_i$ for $n\in\{3,4,7\}$ and nonuniquely minimizes $\sum f_i$ for $n\in\{5,6\}$. However, H_8 has $\sum f_i=24$, whereas $\min\sum f_i=26$ for eight-point subsets of L_Δ. And, at $n=9$, $\min\sum f_i=31$ for L_Δ, whereas the nine-point configuration of Fig. 14 has $\sum f_i=30$. I conjecture for $n=9$ that the Fig. 14 configuration uniquely minimizes $\sum f_i$.

References

[1] P. Erdős, P. Fishburn, Maximum planar sets that determine k distances, Discrete Math. 160 (1996) 115–125.
[2] P. Erdős, P. Fishburn, Distinct distances in finite planar sets, Discrete Math. 175 (1997) 97–132.
[3] H. Harborth, L. Piepmeyer, Three distinct distances in the plane, Geom. Dedicata 61 (1996) 315–327.